THE IMPORTANCE OF TEACHING GÖDEL'S INCOMPLETENESS THEOREM IN MATHEMATICS TEACHER EDUCATION

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ABSTRACT

In this article we discuss our views on the importance of teaching Gödel's incompleteness theorem (GIT) in Mathematics teaching degrees. This theorem addresses formal Mathematics systems, such as Peano arithmetic. It does not directly deal with elementary school contents taught by Mathematics teachers. It highlights a fundamental characteristic of the axiomatic method through which Mathematics is produced. The main ideas established by GIT cover the incompleteness of proof of arithmetic consistency in arithmetical language. Incompleteness means the existence of true propositions about natural numbers that cannot be proved by elementary arithmetic. A corollary established that, if arithmetic is consistent, such consistency cannot be proved by any metamathematical argument that can be represented according to the formalism of arithmetic. In other words, not even the most elementary axiomatic systems of Mathematics, which formalize natural numbers, can prove all truths established in its own spectrum. Knowing this result and the conclusions that it reaches is essential in terms of the mathematical culture of a teacher preparing to teach Mathematics in schools, as a means to avoid nurturing the notion of complete Mathematics and the totalitarianism of the axiomatic method.

INTRODUCTION

Gödel's incompleteness theorem (GIT) underscores one characteristic of Mathematics as a science: there are true propositions obtained from elementary arithmetic of natural numbers that are undecidable using arithmetic language itself. The undecidable, as constructed by Gödel in the demonstration of his theorem, is a true formula, though indemonstrable. The proof of the existence of undecidable propositions shows that there is no algorithm that is able to decide about the veracity or the falsity of *all* mathematical statements, even when these statements are constrained to the language of the theory of natural numbers and the usual operations of sum and multiplication. Moreover, undecidable propositions prove that GIT (and all theories derived from it) is unable to decide over *each and every* sentence that may be expressed using this language.

In addition to the fact that GIT points to the existence of a non-void set of undecidable propositions, it signals the impossibility to demonstrate arithmetic consistency, in arithmetic itself. It is important to highlight that GIT does not prove that all theories are incomplete; rather, it shows that the theories that contain Peano axioms are incomplete in their formalization.

The proof of the arithmetic consistency of natural numbers concerns proof that no theorem derived from arithmetic axioms, which comprise the formal arithmetic, may be shown to be true or false simultaneously.

As we understand it, the importance of this theorem lies in the fact that it declares that there are mathematical truths beyond the reach of the methods resorted to by Mathematics. One important aspect concerns the context in which this result is obtained, when it becomes clear that the subject declaring this statement is arithmetic itself, using mathematical methods. This statement results from the observation of the proof of incompleteness as given by Gödel using the arithmetization of metamathematics, when the undecidable is constructed. Gödel's arithmetization process allows some propositions to express something that is arithmetical in nature at the same time that they express if they are deducible in the system. Such is the case of formula G, which was developed by Gödel and represents the metamathematical proposition "I am not demonstrable". Following that development, the mathematician showed (i) that G is demonstrable if and only if ~G is demonstrable, and (ii) that G is a true formula. Therefore, if G is true and undecidable, then arithmetic's axiomatic system is not complete. If on the one hand formula G expresses this limitation of mathematical expression methods of each and every truth obtained with this theory, on the other it is powerful enough to be aware of and to explain its own limitations.

The presentation and the work with GIT in Mathematics teacher degrees contributes with the formation (form/action) of these teachers, inasmuch as it presents mathematical conclusions that emerge from Mathematics itself, revealing the limits to the human ambition of proving all truths obtained in the scope of a mathematical theory, as well as the ambition that this theory is free of internal contradictions. So, human ambition is withheld when Gödel proves that (i) true propositions are obtained in arithmetic, which are beyond the truths expressed and determined by the formal relationships, and (ii) that demonstrating the consistency of Peano first-order elementary arithmetic axioms requires more powerful theories.

The term formation (form/action) is justified by the fact that the student is absorbed in the drive to become a Mathematics teacher by taking a degree. In this process, the student has to learn about the human and the production of the science he or she will work with, acquiring knowledge about the structure, history, and social relevance of Mathematics and understanding the way the discipline is socially legitimized in the student's own culture as well as in other cultures.

In this article, *formation* is understood as *form/action*, and concerns a continuous process of becoming, when *form* and the *actions involved* are constantly changing, undergoing transformations and yet remain mutually implicated, moving ahead in a process of taking place next to matter, continually being and moving. We understand that formation (form/action) processes are specific to the formation of a person and, therefore, they encompass Education. Bicudo (2010) notes that "formation as conceived with 'projection', that is, understood as the thrownness to which it belongs in the Dasein's state of possibilities that actualize themselves defining a person's styles and modes of being. Formal education narrows the way towards understanding formation, since it is based on a pedagogical project. This project indicates the directions, which are considered positive, for the actualization of the possibilities of people and of those concerning knowledge areas and respective courses. It points to postures, to relationship modes, to procedures that pave the way for teaching and learning" (Bicudo, 2010, p. 216). "Formation (form/action) concerns a game involving the attention with *form and action*, affording the realization of man, who becomes by being, that is, doing". The meaning of formation presents itself as "the idea of pursuing the ideal form, constructed based on the awareness of society's way of life and its yearnings, habits, honor codes, values nurtured as well as the strength that drives people towards the perception of compliance and that makes them feel proud for their deeds. However, the *ideal* is never assumed to be the perfect way that submits *formation* to a model that encapsulates it within unbending boundaries. The ideal is considered to point to the direction of this movement. Yet, this movement takes place with the moving, and the moving also has its own power, which means that *form* cannot conform to *action*; rather, it is *action* that, when it operates with *matter*, gives it its shape.

Regarding the conclusion arrived at by GIT, it does not render the continuation and application of Mathematics unfeasible; however, to know it is to know this science's structure, which is important for the mathematical culture of the professions with a role in teaching it. This theorem highlights, on the one hand, the limitations in formalization in Mathematics, showing that there are truths that cannot be demonstrated; on the other, it signals the importance of the fact that the first-order elementary arithmetic cannot prove its own consistency.

Although GIT is a mathematical result that produces conclusions about its method of production, it also reveals itself as a cultural fact in the scope of Mathematics and of its method to produce truths.

The demonstration of this theorem occurs in the complexity of ideas that could be called theory of incompleteness. Two theorems are highlighted in the proof developed by Gödel. These theorems are usually called Gödel's first and second theorems:

The first incompleteness theorem shows the limitations to the formalization in Mathematics (that is, the notion of formal demonstration sometimes is not robust enough to represent the mathematical notion of truth). The second incompleteness theorem shows that Peano first-order elementary arithmetic, the theories that theories that are equivalent to it or their extensions are not able to demonstrate consistency or, rather, it is not possible to represent a demonstration of consistency in these theories (which requires more robust theories) (Da Silva, 2010, p. 5).

One of the conclusions of GIT concerns the behavior of formalized theories of natural numbers, showing that incompleteness is a characteristic shared by all systems that formalize natural numbers and that this incompleteness uses Peano axioms as non-logical axioms in this formalization. The first incompleteness theorem reveals that we should not nurture the expectation of proving all truths established in arithmetic. In turn, the second theorem declares that the context in which the problem of arithmetic consistency was formulated is not the context in which it can be solved, as discussed by Da Silva (2010).

The incompleteness of a T theory claims that T has a closed formula G, and that neither G nor $\sim G$ are decidable in T. That is, the set of T theorems do not cover all T formulas that are true in arithmetic. In other words, the set of *T* theorems does not correspond exactly to the set of true *T* formulas.

By analyzing the demonstration of GIT, we notice that the undecidable constructed by Gödel does not depend on the formal system considered, as long as the formal logical system is robust enough to contain Peano arithmetic. Since formalization is one of the peculiarities of human knowledge, the result obtained by GIT should be considered every time it is an option. Since natural numbers are the basis of a large number of mathematical theories, incompleteness occurs in all the theories that contain Peano axioms and sum and multiplication operations.

Gödel proved that basic formalized arithmetic, apart from being complete, is incompletable. The notion of incompletable emerges from the fact that if *G*, the *undecidable*, were added to the basis of arithmetic as an ulterior axiom, a new, larger set of axioms would still be insufficient to formally produce all arithmetic truths, since another true arithmetic formula, which would be more undecidable, could be constructed in the new, enlarged system. This new, possible formula would be constructed by repetition of the process conducted to obtain *G* in the new system. This reveals that, even if the system is frequently distended, incompleteness is preserved. Therefore, the undecidable compels us to recognize a fundamental limitation of the power of the axiomatic method: “A set of arithmetic axioms cannot produce, using logics and formal language rules, all true arithmetical proposition, that is, the vast continent of arithmetical truth cannot be taken to a systematic order” (Nagel & Newman, 1973, p. 56).

Considering the message GIT gives in its conclusions, its demonstration, its relevance, and its implications, we understand that it is essential to teach this result to teachers taking degrees that prepare them to teach Mathematics, since the repercussion of this theorem also manifests when it is being taught. The explanation lies in the fact that its impact on the axiomatic system is noticeable, common to all mathematical contents covered in Mathematics curriculums of all levels. Therefore, in order to reveal the ideas in GIT and the aspects it underscores in layers of increasing complexity during the evolution of a teacher education degree, working with GIT is essential to teachers who will teach Mathematics.

Knowing GIT is an important part of the cultural history of Mathematics as a science. This result is a key chapter in mathematical knowledge along history. It is proof that, when unfolded considering its initial development context, expresses the characteristics of the axiomatic method, affording to understand the ambitions that accompany the development of Mathematics and to understand the presence of the human that is directly involved in this development. The mathematical tool to preserve truths from initial truths that have been accepted as such has a flaw that is inherent to its own system. This was explained by GIT, when it shows the presence of true arithmetical formulas that are not obtained by well-defined operations using the initial axioms. According to our understanding, this shows, mainly, the human role in the ambition of Mathematics to dominate the production of truths using axiomatic systems.

In addition, knowing this chapter of the history of Mathematics is a relevant opportunity for us to know ourselves, men and women who spend their lives to craft theories that construct the Mathematics that is legitimized in the essence of western civilization. We understand that this comprehension is important in the development of teachers.

THE MATHEMATICS EDUCATION DEGREES

In Brazil, important studies about Mathematics teacher education have been conducted, especially by each of us who dedicate ourselves to Mathematics Education and have considered this complex, problematic issue for decades.

It is important to clarify the specificity of Mathematics education degrees in Brazil, since this theme is also relevant in other countries, where it is addressed from various perspectives.

Mathematics education degrees in Brazil are higher education degrees that are taught in two approaches: Mathematics *bachelor* degree (to teach professors of Mathematics to work in higher education institutions and to form researchers), and Mathematics *teaching* degree (to form Mathematics teachers for the 6^{th} to the 9^{th} grades of Elementary Schools and the 1^{st} to the 3^{rd} grades of High School). Both approaches follow pedagogical projects for the formation of a Mathematics teaching professional.

In Brazil, teacher education degrees are developed according to decisions CNE/CP01 and 02, published in 2002, which contain the National Curricular Guidelines for the Formation of Elementary School Teachers (CNE/CES, 2001). These guidelines state the independence of bachelor degrees from teaching degrees, in what represents a change from the previous structure, according to which fundamental education training courses covering practical aspects of teaching were offered only in the last year of the teaching degree, forming a Mathematics teacher who would teach grades prior to higher education.

The Mathematics education teaching degrees defined in the legislation last four years. Their pedagogical structures are organized as axes that include the disciplines that cover Mathematics knowledge and similar areas, and Mathematics education and Education.

The structure of Mathematics teaching or bachelor degrees in Brazil prepares students based on projects conceived to develop competencies and skills such as critical posture, logical reasoning, and problem-solving skills, which enable undergraduates to take job positions that are not restricted to school or academic environments.

Mathematics teaching and bachelor degrees differ in their objectives. Teaching degree undergraduates experience a project that addresses formation (form/action) of teachers to teach Mathematics in elementary school. Bachelor degree undergraduates learn contents directed towards formation (form/action) of the mathematician who will teach at higher education organizations or dedicate himself or herself to research. While bachelor degrees are focused on Mathematics contents and prepare undergraduates to produce and understand the production of mathematical knowledge, teaching degrees underscore the development of the sensitivity to enable graduates, that is, Mathematics teachers, to understand the actions of their students, contributing to learning. The major conception is that his teacher perceives and works with the importance of mathematical knowledge for the formation of citizens and understands the importance of mathematical knowledge to everyone, identifying the social, political, and cultural aspects that are observed in this work environment. The main emphasis has been laid on these aspects, so that the structure of mathematical science, as understood by western civilization, is not presented and worked in a way that its ideas and modes of constituting itself are made clear.

As said above, the formation of teachers is different in other countries. While in Brazil Mathematics teaching degrees are obtained in an undergraduate program, in Portugal Mathematics teachers have to take a MSc degree to teach the grades corresponding to elementary and high school in Brazil, as explained by Oliveira and Cyrino (2011). Students have to have a teaching degree initially, and then take a 2-year degree with courses about teaching methodology, Mathematics didactics, and practice of Mathematics teaching, obtaining a degree to teach Mathematics and sciences. This is called initial formation of specialized teachers, and differs from the generalist formation in undergraduate courses, qualifying teachers to teach elementary school, which is equivalent to the 5th grade in Brazil. In Portugal, the formation programs for specialized Mathematics/science teachers prepare undergraduates to teach Mathematics to the last grades of junior high (7th to 9th grades), high school, and higher education.

The Portuguese teacher formation model reflects what happens in European Union countries, as described in the report *Mathematics Teaching in Europe: common challenges and national policies*, EACEA, Eurydice (2011).

Teacher formation is one of the dimensions of Mathematics Education, which prioritizes conceiving and developing projects that take concernment as the mode of becoming of the other, of the teacher in formation, taking care of the becoming of this teaching degree student. In these degrees, "teaching organizes the activities that render viable the effecting of that care, translated in *forms*, *contents*, and *directions* that are addressed" (Bicudo, 1999, p. 5-6, emphasis added). As Mathematics educators, we conceive Mathematics as an end and as a means. As an end, it is an important science in the historical and cultural development of western civilization mainly, from the scientific, technical, and technological standpoints. From this perspective we understand that these teacher formation degrees address their theoretical domains and their tools used to produce mathematical knowledge, as in GIT. As a means, Mathematics is taught while we are face to face with people who are in a formation (form/action) movement. For this reason we are concerned with its results as people and citizens who live in several communities and societies.

GIT IN MATHEMATICS TEACHING COURSES: WHY?

The work with GIT should be the basis of Mathematics teacher formation degrees, since it expresses characteristics of the formal basic arithmetic of natural numbers, which include their incompleteness and its inability to express the demonstration that it is free of any internal contradiction. The proof of GIT clarifies the structure of Mathematics when it shows that this science is developed through axiomatic systems and that even the basic system that involves the Peano axioms in its formalization is not able to prove all truths obtained in the theory of arithmetic. Therefore, we recognize that comprehending the reach of formal systems with which Mathematics is produced is a perception required of teachers. Teaching contents are not affected by this result, just like any other part of Mathematics. However, the conception of Mathematics seen as a science underwent transformations after GIT. The notion of scope of the method with which all Mathematics is produced and the knowledge of Mathematics structure should be one of the central aspects of the pedagogical project of Mathematics teaching degrees.

It is known that, before Gödel, the prevailing conception was that there was no *ignorabimus*[1]. That is, there was no well formulated proposition that, under the right amount of effort by mathematicians, did not have a solution. But with Gödel this concept fails to sustain itself. The meeting between Gödel and the undecidable in his incompleteness theorem lends arithmetic (and the equivalent theories and their developments) the notion that it has mathematically well formulated, true sentences that cannot be proved, which directly affects the idea of the absence of *ignorabimus*.

Therefore, subsequently to Gödel's theorem, it is understood that the theories that include natural numbers in their formalization are formed by propositions that have been demonstrated (theorems), non-demonstrated propositions (conjectures), and undecidable propositions (indemonstrable and irrefutable). Theorems are truths demonstrated in theory. Undemonstrated propositions continue to require solution. Some of these, though, also call for an extension of the theory through which they can be solved. In this case, "solved" means demonstrated as true or false in the theory in question. In addition to theorems and conjectures, there is the proof of the existence of a non-void set of true propositions that cannot be proved as such, in the theory.[2]

In this sense, it can be said that the incompleteness of a theory is a consequence of its axiomatic system.

Since there is an undecidable in a theory, and that the undecidable is a true declaration obtained by a metamathematical argument, it becomes evident that it is a theory that may obtain the truths but cannot prove them all. This means that, in practice, not every proof can be reduced to axioms in the foundation of the system, that is, the axioms in the foundation of the formal system of a specific theory are not a collection that affords to prove all truths in this theory. This also shows that the set of axioms that the undecidable requires to be proved is not clear. For instance, a true sentence in the theory of numbers is possible to be expressed in arithmetical language, that is, it can be derived from Peano axioms, whose proof may resort to topology or complex analysis. In other words, it is possible that the proof of the sentence requires some other axioms of other theory or theories.

It is thus that Gödel declares that Peano axioms describe the theory of natural numbers only in part. So, it is not every consistent body of propositions that can be described by a collection of axioms. GIT states that the body of the natural number theory is a consistent body of propositions without recursive axiomatization. When instructions are computed, a computer can recognize axioms and basic rules to derive theorems and discover whether a proof is valid. However, the computer cannot determine whether a proof to a statement exists, since it is necessary to wait and see if the proof or negation thereof is generated. The result is that this method does not reveal which propositions are theorems. Hence the statement that the axiomatic method suffers from a technical limitation.

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[1] In Hilbert (2003, p.11), it can be read that: "The axiom of solvability of each problem is only a characteristic that is particular to Mathematics thought, or is it possible to create a general law inherent to the nature of our thought that says that all questions framed have answers? […] The conviction that the solvability of a mathematical problem provides a strong stimulus during work, we hear the inner voice shouting, *Da ist das Problem, suche die Lösung. Du kannst sie durch reines Denken finden; denn in der Mathematik gibt es kein Ignorabimus!* (there is the problem, look for the solution. You may find it in pure thought, since in Mathematics there is no 'let's ignore it!')".

[2] A theory is formed by an axiomatic system and all its theorems, which are derived from the set of the axioms in the basis of the theory using an axiomatic method.

GIT confirms the existence of a technical limit to the way Mathematics can be done. We believe that, for this reason, GIT is a cultural result directly associated with the way Mathematics is understood and, therefore, is reflected on the way we teach.

In view of the importance of this theorem, mainly concerning its relevance to Mathematics and Philosophy of Mathematics, we understand that the presentation of the result of the Gödelian phenomenon is essential information and should be included in the list of contents to be taught in Mathematics teaching degrees. Our proposal suggests that this inclusion be carried out in an intellectually fair way, which creates possibilities for the Mathematics undergraduate taking a teaching degree in Brazil to know the outcome of GIT, sensing it already in the first term of a degree and reviewing it in the various dimensions of ideas that it establishes, in addition to its conclusions and its demonstration at different levels throughout the degree, culminating with its formal demonstration (Batistela, 2017).

However, it is important to emphasize that Gödel's theorem is not directly associated with Mathematics contents taught at schools. It is a significant result embedded in the essence of Mathematics' formal method and, in this sense, we understand that it is important that a teacher be qualified to teach Mathematics and know this result so as to clarify the incompleteness of this science and work with this notion.

We understand that knowing the idea of incompleteness is also important from the perspective of the ideology of certainty, which has prevailed in the community of mathematicians and, by extension, in western society. Understanding GIT renders viable to understand the message that says that we should balance any inflated expectation concerning the power of Mathematics concerning producing and proving all its truths in the scope of its theories. Apart from that, this understanding affords to realize the difference between demonstrability and truth in this science.

The mathematical culture of a teacher in formation (form/action) requires the teaching of GIT in its curriculum. It also demands that GIT be taught, since it presents one characteristic of the Mathematics production mode embedded in western civilization, of Mathematics as a whole, and of the Mathematics taught at school, since this reflects the Mathematics worked in teacher formation (form/action) degrees.

The prevailing trend in Mathematics organization, which is a strong trend in Mathematics teaching methods, emerges from the Euclidean method and from that produced by the Bourbaki group. On the one hand, the Euclidean method underscores axiomatics, in which logical and linguistic formal aspects are considered prevalently; the proposal set forward by the Bourbaki group prioritizes the structuralism organized around algebraic, topological, and order structures.

Considering the three philosophical lines that most influenced Mathematics, its production, and, therefore, its teaching, Hilbert's formalism stands out most visibly, though it was also the philosophical line that was most affected by GIT. For Wittman (2001), "Although Hilbert's dream burst already in 1930 when Gödel proved his incompleteness theorem, the formalistic setting of Hilbert's programme has survived and turned into an implicit theory of teaching and learning" (Wittman, 2001, p. 6).

The axiomatic method of a given theory is what crowns its refinement and unveils its adult life. Our interpretation is that GIT reveals one of the characteristics of Mathematics: the incompleteness of part of its theories and, as a result, of Mathematics as a whole. This influences the way through which Mathematics is understood, but it does not influence the way it is produced.

The comprehension that Mathematics answers all its questions is embedded in the ideology of superiority, which is peculiar to those who do not know the impossibility of this science being complete and consistent simultaneously and, as a consequence, of this school subject.

The importance we ascribe to the results of this theorem is in consonance with our view that its reach and its repercussion to Mathematics at school, inasmuch as questions involving social reality, which are usually explained based on mathematical certainty, may be challenged in the foundation of the structure of this science.

We also argue that it is essential to work GIT in courses that form Mathematics teachers, especially due to the view of this science that GIT carries along, shedding light on the possibilities and impossibilities of Mathematics in its effort to produce and prove truths. After 85 years since the GIT was proved, it is unacceptable that Mathematics teachers qualify without knowing this drawback in Mathematics, which dissolves the notion that there is one Mathematics whose theories are able to derive their true propositions, prove all of them, and have them circumscribed to the scope of the axiomatic system that gives them their shape.

GIT IN MATHEMATICS TEACHER DEGREES: WHERE AND HOW?

Below we discuss the aspect of GIT that we believe may be addressed in teaching degrees, in the education model currently adopted in Brazil, together with contents that are taught.

Since it is a result that has been proved in the scope of *Peano arithmetic*, the existence of the Gödelian phenomenon could be mentioned, when this subject is being addressed, revealing one of the aspects of GIT: there is, in arithmetic, a non-void set of true formulas that cannot be proved.

In addition, GIT may be considered when working with the *classical problems of Antiquity* (squaring the circle, angle trisection, doubling the cube), and with Abel's impossibility theorem about the impossibility to solve 5^{th} degree equations with radicals, among others.

We understand the appropriateness of exploring GIT when working with the other theorems of impossibility in Mathematics, presenting it as a theorem that states the impossibility to carry out the demonstration, in arithmetic itself, of the compatibility of arithmetical axioms. It is important to differentiate it from the other impossibility theorems that address the impossibilities associated with the conditions of propositions of problems. The impossibility expressed by the incompleteness theorem is a result that reveals the impossibility to reach a positive solution of Hilbert's problem 2, which implies the impossibility to carry out the original project conceived by formalists.

GIT emerges at a historical moment when Mathematics was the object of attempts towards foundation on different ground. Intuitionistic and logicistic schools, each with their own reasons, had failed when attempting to fully meet the idealized objective. Formalists remained in place, working and adamant towards the group's objective, but Gödel's result, reached in 1931, announces the impossibility of the attempt originally conceived by this school. This subject, *the philosophical schools that aimed to lay the foundation of Mathematics*, is usually addressed in Mathematics teacher formation degrees, and we understand that this would be the moment to present GIT, since it indicates the possibility of laying the foundation of Mathematics fully under the foundations of arithmetic, which was the objective of formalists. In addition, it regulates ambitions and expectations that Mathematics is an absolute science that produces and proves, simultaneously, all truths.

In the thematic axes of the development of teaching degrees that work from the perspective of Mathematics as a human historical-cultural achievement, GIT contributes a view of Mathematics as a science that is subjected to limitations imposed by the characteristics of its production method, the axiomatic method, and/or as a science that lives and happens, still vulnerable to doubts. It may therefore be plausible to introduce the incompleteness of Mathematics as an impossibility, and as an entryway as well. The approach to this result would also be appropriate so as to reveal changes in the way problems are faced in Mathematics after the recognition of GIT. The theorem states that there is a set of indemonstrable problems that does not depend on cognitive ideal of mathematicians and on the effort dedicated to the respective proofs. Though these problems are indemonstrable, they reveal the invigorating nature of Mathematics, which does not allow itself to be circumscribed, which will not have the boundaries of its theories clearly defined, restricted to these limits.

The creation of *non-Euclidean geometries*, since it is a result that addresses the relativity of the mathematical truth in relation to the axioms of the system, shows that such truths are restricted to the axioms of theory, denying the belief that Mathematics is a single, totalizing structure, and becomes similar to the message given by GIT, when it declares that there are mathematical truths that are nod deduced from the axioms of the basis of theory and that cannot be proved in that theory. From this perspective we understand that it is necessary to turn this into an opportunity to approach and discuss the ideas in GIT for teachers in formation (form/action).

It is important to underscore that the comprehension of GIT may occur very gradually and in several layers of articulated information. The inclusion of the demonstration of GIT in an appropriate course requires that the curricular activities carried out prior to that course have exposed ideas, discussed GIT aspects, and worked with important results of mathematical logics that lend support to the comprehension of the details of proof and of the established conclusions, which enable the intelligibility of GIT and of the message it carries. We also understand, apart from the work with the idea of GIT, that the experience with the method of metamathematics arithmetization developed by Gödel and the construction of the undecidable is essential as experience of a Mathematics students.

It is important to present, comment, discuss, and mention GIT whenever it is possible to realize it in Mathematics undergraduate degrees. Professors of different courses could (and it would be desirable that they actually did!) present GIT in different dimensions, perspectives, and objectives, especially when they organize to address the themes listed in curriculums.

In a specific course that covers GIT, we suggest the work with the demonstration based on Nagel and Newman (1973)[3], which presents an illustration of the demonstration carried out

[3] Aware of the objectives of this article, as declared by its authors, proposing an approximation of GIT, with the freedom to address GIT in a broad sense, being a technical proof with explanation of details, without the concern to be a formal demonstration of Gödel's theorem.

by Gödel in 1931. It is through this work that we believe it possible to afford students the intelligibility of ingenious ideas and the processes used by Gödel in his original demonstration, as well as the ideas involved in it and of its conclusions.

We believe it appropriate to also present the study of GIT in its formalized design, in a course that presents Mathematical Logic as a theory of formal systems and as the foundation of Mathematics. Therefore, this result affords to address a formal demonstration. We suggest the demonstration based on Shoenfield (1967), which is an alternative demonstration of GIT that, remarkably, is different from the original demonstration produced by Gödel, and is the validation of GIT in Mathematics that, objectively speaking, dedicates itself to present the result of a formal, direct version, distant from discussions and additions.

We understand that it is important that students taking teaching degrees know GIT, understanding the ideas that may be discussed considering the aspects it reveals and the demonstration of this result, so that they may know the architecture of its proof.

In this investigative effort we declare that the characteristics of mathematical formalization should be the focus of concern in bachelor and teaching Mathematics degrees. We also state that working with GIT and its interpretations is important, or even crucial, to understand both the structure of Mathematics, as the instability implicit to formal systems. However, we should make it clear that there are several other demonstrations present in classical books on Mathematical Logic that could allow knowing the development of the proof and the establishment of conclusions.

UNDERSTANDING THE IMPORTANCE OF TEACHING GIT IN MATHEMATICS TEACHING DEGREES

We understand that there are several possible levels in working with GIT with Mathematics undergraduates, underlining the model of teaching degrees according to the current Brazilian legislation. It is possible to address the discussion of the proposition without covering the details of proof and also address a formal demonstration that includes details, exploring it in connection with the ideas it articulates. It is also possible, for instance, to present it as a result that establishes points of contact with other articulated themes, or that may be articulated, with a demonstration of this result or with the discussion of the judgement it formulates.

Therefore, we believe it appropriate to bring up GIT and explore it in its ideas whenever the following topics are being highlighted: formal Mathematics systems, Peano arithmetic, classical Antiquity problems, Mathematics philosophical schools, non-Euclidean geometries, Mathematics as a human historical-cultural realization. GIT should also be invoked and explored when working with two demonstrations of this theorem in courses that present the mathematical structure and logic as formal theory. It is important to consider that one teaching approach that addresses all these mathematical details realized in the demonstration of this theorem requires more time than the usual duration of a course. We believe that, due to teaching hours, as usually adopted in undergraduate degrees, this is not possible, not even in Mathematics bachelor degrees that offer mathematical courses, since these teaching hours would still be not enough to carry out an all-encompassing and in-depth work about the ideas mobilized in GIT proof and the consequences it establishes.

However, even though we understand the difficulty to master the technique to prove the incompleteness theorem, we believe that the understanding of the meanings and significations of GIT already contributes with formal thought and widens the horizons to carry out the work with intuitions, affording experience with the demonstration process.

It is essential that this theme be addressed in the Mathematics Education of future Mathematics teachers, independently of the grades they will be teaching, since the theorem presents the dimension of the reach Mathematics may produce and, for this reason, it is related with the foundations of this science. We understand that the experience of Mathematics teaching undergraduates with this theorem may improve the comprehension of GIT in its various aspects and, mainly, in what concerns the notion of Mathematics. Experience with GIT is the way we visualize students getting to know this result, mathematically. It is thus that the proposal enables teaching degree undergraduates to realize the incompleteness of Mathematics and what it means, mainly to Mathematics itself, to mathematicians, and to Mathematics teachers in elementary school.

We believe that teaching GIT may lead teaching degree undergraduates to realize that there is an incompleteness in the science he or she works with, which however does not prevent the continuity of its production and, as a teaching subject, reveals the structure of Mathematics, since the GIT result disagrees with the consolidated and widely known expectation: that Mathematics would solve each and every question in its own scope, offering also the certainty concerning topics that it covers, in terms of applicability.

Understanding Mathematics beyond what is taught to students is something that affords a wide view of this science. Metaphorically speaking, this statement is similar to that about an airplane pilot having knowledge that passengers are not required to have, even though everyone is on the same plane. The pilot needs to know, for instance, fuel capacity and the combustion rate of engines per distance covered. The teacher has to know more than the characteristics of what he or she is teaching: it is necessary to understand the ideas underlying the mathematical contents covered and those that transcend the objective of formalized Mathematics.

The incompleteness of mathematical theories that include arithmetic do not render unfeasible the continuation of Mathematics as a science, nor does it destroy a number of the theories comprising it. This is a result that influences the conception of Mathematics, not the mathematical doing.

We consider essential that Mathematics bachelor undergraduates know GIT due to the importance of the result and also because, although the projects of these degrees aim at forming Mathematics researchers, these professionals are often hired as professors at the departments of Mathematics to teach courses addressing specific knowledge and courses shared by bachelor and teaching degrees, to future researchers and teachers.

It is thus that our proposal to include GIT in courses taught at teaching degrees requires from teachers who took a bachelor degree to pay attention to the work with GIT in the courses they teach and, whenever possible, to articulate with specific contents and/or the ideas addressed in that component.

REFERENCES

BATISTELA, R. F. (2017). O *Teorema da Incompletude de Gödel em cursos de licenciatura em Matemática*. 2017, 140 f. Tese de Doutorado em Educação Matemática. Rio Claro: Universidade Estadual Paulista "Júlio de Mesquita Filho". Acesso em 9 dez. 2017. Disponível em: <https://repositorio.unesp.br/bitstream/handle/11449/148797/batistela_rf_dr_rcla.pdf?sequence=3>.

Bicudo, M. A. V. (2010). (Ed.). *Filosofia da Educação Matemática: Fenomenologia, concepções, possibilidades didático-pedagógicas*. São Paulo: Editora UNESP.

Bicudo, M. A. V. (1999). Ensino de matemática e educação matemática: algumas considerações sobre seus significados. *Bolema, Rio Claro, 13*, 1-11.

CNE/CES (2001) *Diretrizes Curriculares Nacionais para os Cursos de Matemática, Bacharelado e Licenciatura.* Online: http://portal.mec.gov.br/cne/arquivos/pdf/CES13022.pdf>.

Da Silva, J. J. (2010). *O método axiomático. Curso de verão.* Online: http://docslide.com.br/documents/historia-do-metodo-axiomatico.html.

EACEA/Eurydice (2011) *O Ensino da Matemática na Europa: Desafios Comuns e Políticas Nacionais.* Brussels: EACEA/Eurydice.

Gödel, K. (1977). Acerca das Proposições Formalmente Indecidíveis dos Principia Mathematica e Sistemas Correlatos. In M. Lourenço, (Ed.). *O Teorema de Gödel e a Hipótese do Contínuo* (pp. 245-290). Lisboa: Fundação Calouste Gulberkian.

Hilbert, D. (2003) Problemas matemáticos. *Revista Brasileira de História da Matemática. (Translated by Nobre, S), 3*(5), 5-12.

Imenes, L. M. P. (1989). Um Estudo sobre o Fracasso do Ensino e da Aprendizagem da Matemática. MSC dissertation, Universidade Estadual Paulista, Rio Claro.

Nagel, E., & Newman, J. R. (1973). *Prova de Gödel. (Translated by Guinsburg, G. K.)*. São Paulo: Editora Perspectiva e Editora da Universidade de São Paulo.

Oliveira, H., & Cyrino, M. (2011). Formação inicial de professores de Matemática em Portugal e no Brasil: Narrativas de vulnerabilidade e agência. *Interacções 18*, 104-130.

Shoenfield, J. R. (1967). *Mathematical Logic*. London: London Publishing Company.

Wittmann, E. C. H. (2001). Developing Mathematics Education in a Systemic Process. *Educational Studies in Mathematics 48*(1), 1-20. Online: http://www.jstor.org/stable/3483113.